Topics in complex analysis   Lecture 7   (Nov 4, 2024)

Theorem 5.4 (Bloch's theorem). *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $f'(0) = 1$. Then there exists $p \in \mathbb{C}$ such that $B_{\frac{3}{2}-\sqrt{2}}(p) \subset f(B_1(0))$.*

Proof   Step 1   We claim: $\mathcal{U} \subseteq \mathbb{C}$ bdd domain $g \in \mathcal{H}(\bar{\mathcal{U}})$ not cst.

and $a \in \mathcal{U}$ is st $s := \inf_{z \in \partial \mathcal{U}} |g(z) - g(a)| > 0$ (Assumption

then $B_s(g(a)) \subseteq g(\mathcal{U})$.

Proof of Step 1   By bddness of g, $\partial g(\mathcal{U})$ is compact (closed, bdd)

Thus $\exists\, w \in \partial g(\mathcal{U})$ st   cty of distance fct

$$\mathrm{dist}\left(\partial g(\mathcal{U}), g(a)\right) = |w - g(a)|$$

We claim $|w - g(a)| \geq s$ hence Step 1 is shown

$B_s g(a) \subseteq g(\mathcal{U})$

By def. of $\partial g(\mathcal{U})$ $\exists (z_n)$ seqn in $\mathcal{U}$ st $g(z_n) \xrightarrow{(2)} w$

wlog (up to passing to a subseqn) assume $z_n \xrightarrow[(1)]{} \exists\, z \in \bar{\mathcal{U}}$

(z_n) seqn in a bdd set hence subconverges by Bolzano-Weierstraß

By continuity $g(z) \overset{(1)}{=} \lim_{n \to \infty} g(z_n) \overset{(2)}{=} w \in \partial g(\mathcal{U})$.

By the open mapping thm necessarily $z \in \partial \mathcal{U}$. Hence z

is an admissible pt in the above infimum. Hence

$$|w - g(a)| = |g(z) - g(a)| \geq s.$$

<u>Step 2</u> $\quad g \in \mathcal{H}(\bar{B}_r(a))$ nonconst and $\displaystyle \underbrace{\sup_{z \in B_r(a)} |g'|(z) \le 2|g'(a)|}_{\text{Assumption}}$

then

$$B_R(g(a)) \subseteq g(B_r(a)), \quad R := (3 - 2\sqrt{2})\, r\, |g'(a)|$$

Proof of Step 2. wlog assume $g(a) = a = 0$ (otherwise consider $z \longmapsto g(z+a) - g(a)$)

Set $A(z) = g(z) - g'(0) z$ (first order Taylor polynomial)

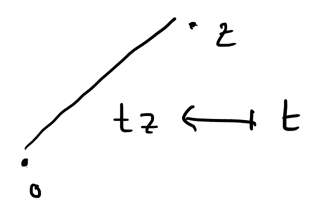
$$A(z) = \underbrace{\int_{[0,z]} g'(\xi)}_{g(z) - g(0)} \underbrace{- g'(0)\, d\xi}_{-g'(0)(z-0)} \qquad (g(a) = a = 0)$$

fund thm of calc

Goal: find appr upper & lower bds on A. upper:

$$|A(z)| \le \int_0^1 \underbrace{|g'(tz) - g'(0)|}_{?} |z|\, dt$$

$$\left(\text{because e.g. } \int_{[0,z]} g'(\xi)\, d\xi = \int_0^1 g'(tz)\, z\, dt\right)$$



By Cauchy Integral formula

$$g'(v) - g'(0) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\xi)}{\xi - v} - \frac{g'(\xi)}{\xi}\, d\xi$$

$$\overset{\text{algebra}}{=} \frac{v}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\xi)}{\xi(\xi - v)}\, d\xi \qquad (v \in B_r(0))$$

$g'(\xi) \le \sup_{z \in \partial B_r(0)} |g'|$

Taking the absolute value results in

$$|g'(v) - g'(0)| \leq \sup_{z \in B_r(0)} |g'(z)| \frac{|v|}{r - |v|}$$

Reinserting this into 2

$$|A(z)| \leq \int_0^1 \frac{|tz|}{r - |tz|} \sup_{z \in B_r(0)} |g'(z)| \, |z| \, dt$$

Assumption

Basic integr $\to$ $= \frac{1}{2} \frac{|z|^2}{r - |z|} \sup_{z \in B_r(0)} |g'(z)| \leq \frac{|z|^2}{r - |z|} |g'(0)|$

Lower Bound: By def of $A(z) = g(z) - g'(0) z$

$$|A(z)| \geq |g'(0)| \, |z| - |g(z)| \quad \text{(triangle ineqn)}$$

Rearranging the ineqn btw the orange parts:

$$|\underbrace{g(z) - g(0)}_{=0}| \geq \left(|z| - \frac{|z|^2}{r - |z|} \right) |g'(0)|$$

where does the term in brackets become maximal?

[The fct $\rho \longmapsto \rho - \frac{\rho^2}{r - \rho}$ becomes maximal on $(0, r)$

at $\rho^* = \left(1 - \frac{1}{2}\sqrt{2}\right) r$ with $(3 - 2\sqrt{2}) r$]

This is where the radius from Bloch's thm comes from

Odd done

Apply step 1 to $\mathcal{U} = B_{\rho^*}(0)$

and $a = g(a) = 0$ to get $B_R(0) \subseteq g(B_r(0))$

$$s = \inf_{z \in \partial B_{\rho^*}(0)} |g(z) - \underbrace{g(0)}_{=0}| \geq \inf_{z \in \partial B_{\rho^*}(0)} \left[|z| - \frac{|z|^2}{r - |z|} \right] |g'(0)|$$

const $= (3 - 2\sqrt{2}) r$

Step 3 conclusion

Proof of Step 3. Given $f \in \mathcal{H}(\overline{B}_1(0))$ consider

$$h(z) = |f'(z)|\,(1-|z|)$$

cts on $\overline{B}_1(0)$

Hence h has a maximum on $\overline{B}_1(0)$ (call it $p \in \overline{B}_1(0)$)

$$M = h(p) \geq h(0) = |f'(0)| = 1 \quad (*)$$

Hypothe of Bloch

Def $r = \frac{1}{2}(1-|p|)$ st $M = 2r|f'(p)|$

"$h(p) = M$"

and $B_r(p) \subseteq B_1(0)$

triangle inequ.

Moreover $\forall z \in B_r(p)$ by triangle inequ $|z| \leq |p| + r = 1-r$

equiv $1-|z| \geq r$. Since by maximality

$$|f'(z)|\,\underbrace{(1-|z|)}_{\geq r} \leq 2r|f'(p)|$$

Def of r.

$\Longrightarrow$

cancel r's $\displaystyle\sup_{z\in B_r(p)} |f'(z)| \leq 2|f'(p)|$

Step 2 $\Longrightarrow$ $B_R(f(p)) \subseteq f(B_1(0))$

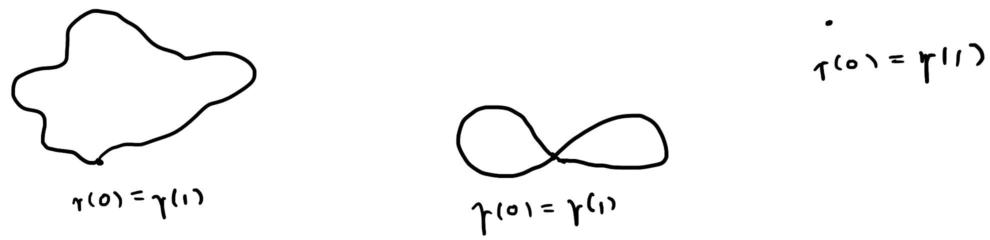
for $R = (3-2\sqrt{2})\,r\,|f'(p)|$

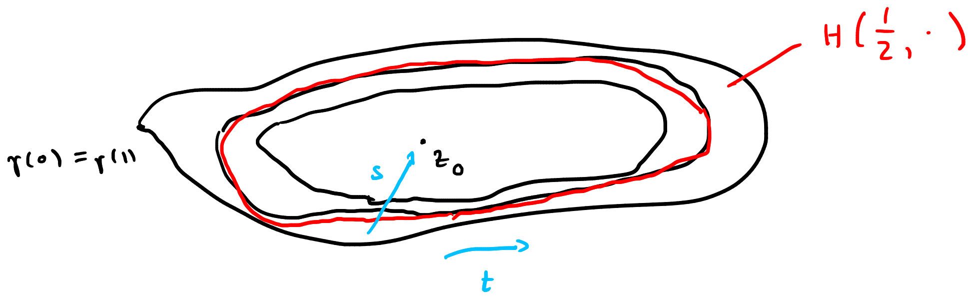
def of r $\longrightarrow$ $= \left(\frac{3}{2}-\sqrt{2}\right)\underbrace{(1-|p|)\,|f'(p)|}_{= h(p) = M \geq 1 \ (*)} \geq \frac{3}{2}-\sqrt{2}.$ $\square$

Definition 5.6. Let $G \subset \mathbb{C}$ be an open set. We say that G is simply connected if it is path-connected and every closed curve $\gamma \subset G$ can be contracted in G to a point, that is, for every continuous curve $\gamma : [0,1] \to G$ with $\gamma(0) = \gamma(1)$ there exists a point $z_0 \in G$ and a continuous map $H : [0,1] \times [0,1] \to G$ such that

"Homotopy" (topology)

(i) $H(0,t) = \gamma(t) \qquad \forall t \in [0,1]$;
(ii) $H(1,t) = z_0 \qquad \forall t \in [0,1]$;
(iii) $H(s,0) = H(s,1) \qquad \forall s \in [0,1]$.





Lemma 5.9. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{-1,1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that*

$$f = \cos(h).$$

Proof The function $z \longmapsto 1 - f^2(z)$ is never zero on G (because $f(G)$ doesn't contain 1 and -1). By Cor 5.8 there exists a holomorphic sqrt $g = \sqrt{1-f^2}$. Note

$$1 = f^2 + g^2 = (f + ig)(f - ig) \qquad (*)$$

Hence $f + ig$ is never zero on G and hence (again by Cor 5.8) $f + ig = e^{ih}$, $h : G \longrightarrow \mathbb{C}$ holo

$h = \frac{1}{i} \log(f + ig)$

By $(*)$ this means

$$f - ig = e^{-ih}$$

$$\Longrightarrow f = \frac{1}{2}(e^{ih} + e^{-ih}) = \cos(h).$$

□